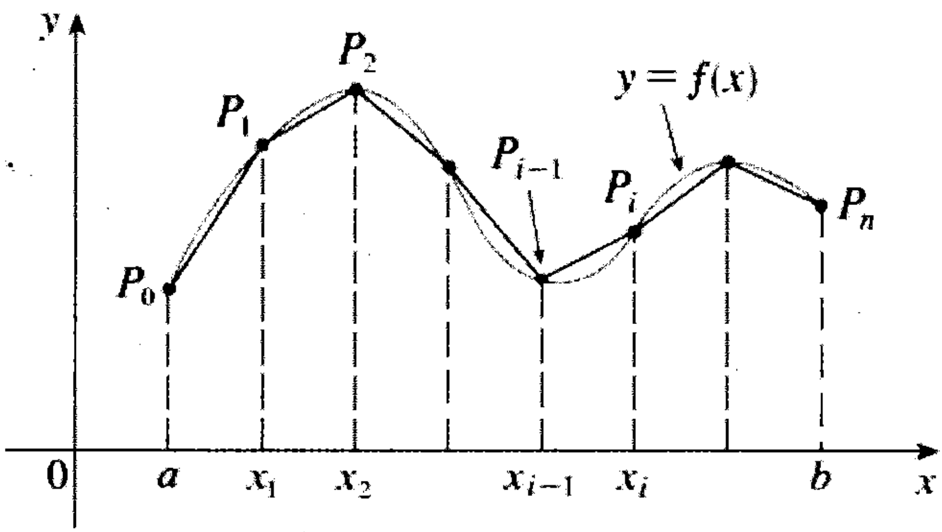
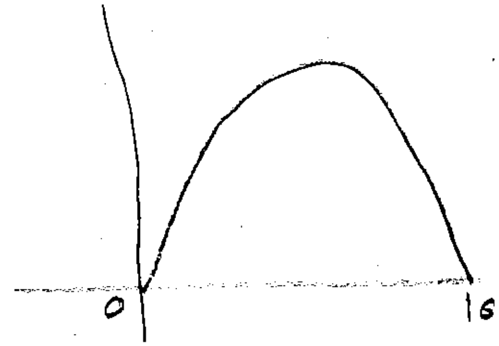
3.1 Arc Length

Goal: Given y = f(x) from x = a to x = b.
Want to find the ***length*** along the curve.

$$\text{Arc Length} = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx$$



$\boxed{\text{Ex}}$



$y = -x^2 + 10x$

HOW LONG IS THIS CURVE?

$y' = -2x + 10$

$$\int_0^{10} \sqrt{1 + (-2x+10)^2}\,dx$$

$$= \int_0^{10} \sqrt{1 + 4x^2 - 40x + 100}\,dx$$

$$\int_0^{10} \sqrt{4\left(x^2 - 10x + \tfrac{101}{4}\right)}\,dx$$

$$2\int_0^{10} \sqrt{x^2 - 10x + 25 - 25 + \tfrac{101}{4}}\,dx$$

$$2\int_0^{10} \sqrt{(x-5)^2 + \tfrac{1}{4}}\,dx$$

$x - 5 = \frac{1}{2}\tan\theta$ ...... $\approx \boxed{51.7485}$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

Derivation:

1. Break into n subdivision:
   $\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$
2. Compute $y_i = f(x_i)$.
3. Compute the straight line distance from (x_i, y_i) to (x_{i+1}, y_{i+1}).

$$\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

$$= \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y_i)^2}{(\Delta x)^2}\right)}$$

$$= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\ \Delta x$$

4. Add these distances up.

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\ \Delta x$$

Note that

$$\lim_{\Delta x \to 0} \frac{\Delta y_i}{\Delta x} = \text{slope of tangent} = f'(x)$$

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(f'(x)\right)^2}\ \Delta x$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Good news:
Arc length is important. And we found an integral to compute arc length, yeah!

Bad news:
The arc length integral almost always is something that can't be done explicitly (we have to approximate, *Simpson's rule*), boo!

In homework, you see the few, *unusual* cases where you actually can compute arc length explicitly.

Here are most of the 8.1 HW questions
Find the arc length of

1. $y = 4x - 5$ for $-3 \leq x \leq 2$.

2. $y = \sqrt{2 - x^2}$ for $0 \leq x \leq 1$.

3. $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ for $1 \leq x \leq 2$.

4. $y = \dfrac{1}{3}\sqrt{x}(x - 3)$ for $4 \leq x \leq 16$.

5. $y = \ln(\cos(x))$ for $0 \leq x \leq \pi/3$.

6. $y = \ln(1 - \mathrm{x}^2)$ for $0 \leq x \leq 1/7$.

Example:

$$y = 4x - 5 \text{ for } -3 \le x \le 2.$$

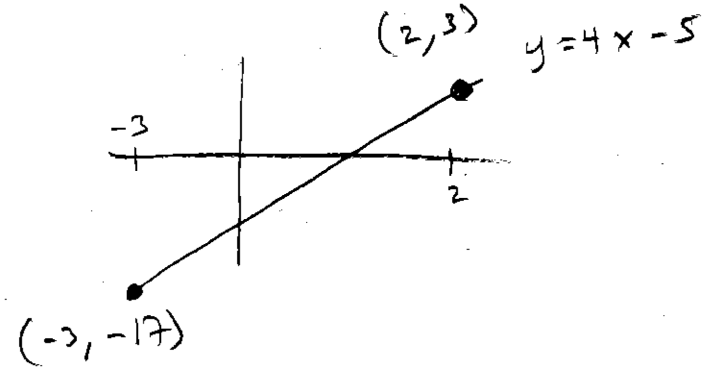
$y' = 4$

$$\int_{-3}^{2} \sqrt{1 + (4)^2}\, dx$$

$$\sqrt{17} \times \Big|_{-3}^{2}$$

$$\sqrt{17}\,(2 - -3)$$

$$\boxed{5\sqrt{17}}$$



$$\sqrt{(2 - -3)^2 + (3 - -17)^2}$$

$$\sqrt{25 + 400}$$

$$\sqrt{425}$$

$425 = 25 \cdot 17$

Example:

$$y = \frac{x^4}{8} + \frac{1}{4x^2} \quad \text{for } 1 \le x \le 2$$

$$\frac{1}{8}x^4 + \frac{1}{4}x^{-2}$$

$$\left(\tfrac{1}{2}x^3 - \tfrac{1}{2}x^{-3}\right)\left(\tfrac{1}{2}x^3 - \tfrac{1}{2}x^{-3}\right)$$

$$y' = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$$

$$1 + \boxed{\left(\tfrac{1}{2}x^3 - \tfrac{1}{2}x^{-3}\right)^2}$$

$$= 1 + \boxed{\tfrac{1}{4}x^6 - \tfrac{1}{2} + \tfrac{1}{4}x^{-6}}$$

$$= \tfrac{1}{4}x^6 + \tfrac{1}{2} + \tfrac{1}{4}x^{-6}$$

$$= \left(\tfrac{1}{2}x^3 + \tfrac{1}{2}x^{-3}\right)^2$$

$$\int_1^2 \sqrt{1 + \left(\tfrac{1}{2}x^3 - \tfrac{1}{2}x^{-3}\right)^2}\, dx \quad ???$$

$$= \int_1^2 \sqrt{\left(\tfrac{1}{2}x^3 + \tfrac{1}{2}x^{-3}\right)^2}\, dx$$

$$= \int_1^2 \tfrac{1}{2}x^3 + \tfrac{1}{2}x^{-3}\, dx$$

$$= \tfrac{1}{8}x^4 - \tfrac{1}{4}x^{-2}\Big|_1^2$$

$$= \left(\tfrac{1}{8}(2)^4 - \tfrac{1}{4}\tfrac{1}{(2)^2}\right) - \overbrace{\left(\tfrac{1}{8} - \tfrac{1}{4}\right)}^{-1/8}$$

$$= 2 - \tfrac{1}{16} + \tfrac{1}{8} = \frac{32 - 1 + 2}{16} = \boxed{\frac{33}{16}}$$

Example:

$y = \ln(\cos(x))$ for $0 \le x \le \pi/3$.

$$y' = \frac{1}{\cos(x)}(-\sin(x)) = -\tan(x)$$

$$\sin(\pi/3) = \sqrt{3}/2$$
$$\cos(\pi/3) = 1/2$$
$$\sin(0) = 0$$
$$\cos(0) = 1$$

$$\int_0^{\pi/3} \sqrt{1 + (-\tan(x))^2}\, dx$$

$$\int_0^{\pi/3} \sqrt{1 + \tan^2(x)}\, dx$$

$$\int_0^{\pi/3} \sqrt{\sec^2(x)}\, dx$$

$$\int_0^{\pi/3} \sec(x)\, dx$$

$$= \ln|\sec(x) + \tan(x)| \Big|_0^{\pi/3}$$

$$= \ln|\sec(\pi/3) + \tan(\pi/3)| - \ln|\sec(0) + \tan(0)|$$

$$= \ln|2 + \sqrt{3}| - \ln|1 + 0|$$

$$= \boxed{\ln(2 + \sqrt{3})}$$

side (don't need all this for this course)

n applications, Arc Length is used in notion (parametric) problems, which you vill see a lot in Math 126:

$$x = x(t), y = y(t)$$

n this case, the same derivation from the eginning of class yields:

$$\text{rc Length} = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

his gives the ***distance*** the object has raveled on the curve.

Very often, in motion problems we need:

$$s(t) = \int_0^t \sqrt{\left(x'(u)\right)^2 + \left(y'(u)\right)^2}\, du$$

which gives the distance traveled from time 0 to time t. This is called the ***Arc Length (Distance) Function***.

imple Example:

onsider

$$x = 3t, y = 4t + 2$$

where t is in seconds.

(a) Find the arc length from 0 to 10 sec.
(b) Find the arc length function.
(c) What is the derivative of the arc length function?

(a) $x' = 3, \quad y' = 4$

$$\int_0^{10} \sqrt{(3)^2 + (4)^2}\, dt = \int_0^{10} 5\, dt$$
$$= 5t \Big|_0^{10}$$
$$= 50$$

b) $\int_0^t \sqrt{(3)^2 + (4)^2}\, du$

$$5u\Big|_0^t = 5t$$

$$\boxed{s(t) = 5t}$$

$$s(t) = \int_0^t 5\, dt$$
$$= 5t = \text{distance}$$

$\boxed{s'(t) = 5}$ ft/sec ← Sp...

$$s'(t) = \sqrt{(x'(t))^2 + (y'(t))^2}$$